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Abstract. Phase Space is the framework best suited for quantizing superintegrable systems—systems with more conserved quantities than degrees of freedom. In this quantization method, the symmetry algebras of the hamiltonian invariants are preserved most naturally, as illustrated on nonlinear σ-models, specifically for Chiral Models and de Sitter N-spheres. Classically, the dynamics of superintegrable models such as these is automatically also described by Nambu Brackets involving the extra symmetry invariants of them. The phase-space quantization worked out then leads to the quantization of the corresponding Nambu Brackets, validating Nambu's original proposal, despite excessive fears of inconsistency which have arisen over the years. This is a pedagogical talk based on [1,2], stressing points of interpretation and care needed in appreciating the consistency of Quantum Nambu Brackets in phase space. For a parallel discussion in Hilbert space, see T Curtright's contribution in these Proceedings, [hep-th/0303088].

Deformation Quantization of Nambu Mechanics

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I INTRODUCTION

Highly symmetric quantum systems are often integrable, and, in special cases, superintegrable and exactly solvable [3]. A superintegrable system of N degrees of freedom has more than N independent invariants, and a maximally superintegrable one has $2N-1$ invariants. The classical evolution of all functions in phase space for such systems is alternatively specified through Nambu Brackets (NB) [4–7]. However, quantization of NBs has been considered problematic ever since their inception. We find that it need not be.

In the case of velocity-dependent potentials, when quantization of a classical system presents operator ordering ambiguities involving x and p, the general consensus has long been [8,9,?,?] to select those orderings in the quantum hamiltonian which maximally preserve the symmetries present in the corresponding classical hamiltonian. Even for simple systems, such as σ-models considered here, such constructions may become involved and needlessly technical.

There is a quantization procedure ideally suited to this problem of selecting the quantum hamiltonian which maximally preserves integrability. In contrast to conventional operator quantization, this problem is addressed most cogently in Moyal's phase-space quantization formulation [12,13], reviewed in [14]. The reason is that the variables involved in it ("phase-space kernels" or "Weyl-Wigner inverse transforms of operators") are c-number functions, like those of the classical phase-space theory, and have the same interpretation, although they involve $\hbar$-corrections ("deformations"), in general—so $\hbar \to 0$ reduces to the classical expression. It is only the detailed algebraic structure of

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